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Singular Solutions of Nonlinear Differential Equations — an application of Fuchsian differential equations

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1 Introduction

We consider nonlinear partial differential equations of Kovalevskian type

$$\partial_t^m u = f\Big(t, x; (\partial_t^j \partial_x^\alpha u)_{\substack{j\leqslant m-1 \\ j+|\alpha|\leqslant m}}\Big) \tag{1}$$

where $t \in \mathbb{C},\ x \in \mathbb{C}^d$ and the coefficients are holomorphic in a neighborhood Ω of the origin in $\mathbb{C}^{d+1}$.

We give a simple example to explain the motivation, before introducing complicated notations.

Example 1 (Burgers equation).

$$u_{tt} + 2uu_t - u_x = 0 \tag{2}$$

has a formal Laurent series solution

$$u = t^{-1} + gt + \big(\frac{1}{10}g_x - \frac{1}{5}g^2\big)t^3 + \cdots , \tag{3}$$

where $g = g(x)$ is an arbitrary holomorphic function.

It is easy to obtain such a formal solution (3): First, assume u is of the form

$$u = t^\sigma \sum_{n=0}^{\infty} u_n(x) t^n \qquad (u_0 \neq 0), \tag{4}$$

substitute (4) into the equations and then equate the coefficients of the power of t to 0. We have $\sigma = -1$, $2u_0(1-u_0) = 0$ and

$$(n+1)(n-2)u_n = -2 \sum_{\substack{i+j=n \\ 1\leqslant i,j\leqslant n-1}} (j-1)u_i u_j + u_{n-2,x} \quad (n \geqslant 1). \tag{5}$$

It is natural to ask whether the formal series (3) converges or not. Of course, it converges to define exact solutions, which are **singular** on $t = 0$. We have four proofs of its convergence.

(I) Linearization:

By setting

$$u = (\ln w)_t = \frac{w_t}{w},$$

Burgers equation is equivalent to the linear equation

$$w_{tt} - w_x = 0. \tag{6}$$

Singular solution (3) is given by the initial condition

$$w|_{t=0} = 0, \qquad w_t|_{t=0} = h(x)$$

with a suitably chosen $h(x)$.

(II) Direct estimates:
Using recurrent equation (5), Ishii [2] and Ōuchi [4] estimeted the u_n's directly.

(III) Leray-Volevich system:
Let $u = \frac{1}{\lambda}$, then λ satisfies the equation

$$\lambda\lambda_{tt} + 2\lambda_t - 2\lambda_t^2 - \lambda\lambda_x = 0. \tag{7}$$

After some calculation, Equation (7) reduces to the following Leray-Volevich system

$$\begin{cases} \lambda_t = 1 - \lambda\lambda_x + \lambda^2\mu, \\ \mu_t = \lambda\mu_x + \lambda_x\mu - \lambda_{xx}. \end{cases} \tag{8}$$

The solution (3) is given by the initial condition

$$\begin{cases} \lambda|_{t=0} = 0, \\ \mu|_{t=0} = h(x) \end{cases}$$

with suitably chosen $h(x)$.

(IV) Fuchsian differential equations:
Let $\sigma = -1$ and put

$$a_N = \sum_{n=0}^{N} u_n t^{n+\sigma}, \quad w_N = \sum_{n=0}^{\infty} u_{N+1+n} t^{n+1}. \tag{9}$$

Then $u = a_N + t^{N+\sigma} w_N$ and

$$a_{N,tt} + 2a_N a_{N,t} - a_x = t^{\sigma-2+N+1} \times H_N \tag{10}$$

where H_N is a holomorphic function. Substituting (9) into (2), we obtain from (10)

$$\begin{aligned} &(t\partial_t + N + 1)(t\partial_t + N - 2)w_N = \\ &\qquad tA_N + tB_N w_N + tC_N(t\partial_t + N + \sigma)w_N - t^2 v_{N,x} + 2t^N w_N (t\partial_t + N + \sigma)w_N, \end{aligned} \tag{11}$$

where A_N, B_N and C_N are some holomorphic functions. Now we can apply a theorem by Gérad–Tahara [1].

Consider the following nonlinear differential equation:

$$\left(t\partial_t\right)^m w = F\left(t, x; \left((t\partial_t)^j \partial_x^\alpha w\right)_{(j,\alpha)\in\Lambda}\right), \tag{12}$$

where $F(t,x;Z)$ is holomorphic in a neighborhood of $(t,x;Z)=(0,0;0)$ and satisfies

$$F(0,x;0)\equiv 0, \tag{13}$$

$$\frac{\partial F}{\partial Z_{j,\alpha}}(0,x;0)\equiv 0 \qquad \text{if } |\alpha|>0. \tag{14}$$

The characteristic polynomial of (12) is

$$C(\rho,x):=\rho^m-\sum_{j=0}^{m-1}\frac{\partial F}{\partial Z_{j,0}}(0,x;0)\rho^j. \tag{15}$$

Theorem 1 (Gérard-Tahara). *If $C(n,0)\neq 0$ for all positive integers n, then* (12) *has a unique formal solution $w=\sum_{n=1}^{\infty}w_n(x)t^n$ with $w(0,x)\equiv 0$, where $w_n(x)$ are holomorphic on a common neighborhood of the origin in $\mathbb{C}^d$. Moreover this power series is convergent and holomorphic near the origin in $\mathbb{C}_t\times\mathbb{C}^d_x$.*

2 Characteristic Exponent

We put

$$\Lambda:=\{(j,\alpha)\in\mathbb{N}\times\mathbb{N}^d:\ j<m,\ j+|\alpha|\leqslant m\}$$

and write (1) as

$$\partial_t^m u=f\left(t,x;\partial^\Lambda u\right), \tag{16}$$

where $f(t,x;Z)$ is holomorphic in $\Omega\times\mathbb{C}^{\#\Lambda}$.

We expand f in Z

$$f(t,x;Z)=\sum_{\mu\in\mathcal{M}}f_\mu(t,x)Z^\mu, \tag{17}$$

where $\mathcal{M}$ is a subset of $\mathbb{N}^{\#\Lambda}$.

Let $k_\mu\in\mathbb{N}$ be the valuation of $f_\mu(t,x)$ in t,

$$f_\mu(t,x)=t^{k_\mu}\sum_{k=0}^{\infty}f_{\mu,k}(x)\,t^k. \tag{18}$$

Definition 1. The characteristic exponent σ_c of (16) with respect to the surface $t=0$ is

$$\sigma_c:=\sup_{\substack{\mu\in\mathcal{M}\\ |\mu|\geqslant 2}}\frac{\gamma_t(\mu)-m-k_\mu}{|\mu|-1}, \tag{19}$$

where

$$|\mu|:=\sum_{(j,\alpha)\in\Lambda}\mu_{j,\alpha},\qquad \gamma_t(\mu):=\sum_{(j,\alpha)\in\Lambda}j\cdot\mu_{j,\alpha}.$$

We assign weights as follows:

$$u\to\sigma\qquad \partial_t\to -1\qquad t\to 1.$$

Then the total weight of the right hand side of (16) is $m-\sigma$ and that of the term $f_\mu(\partial^\Lambda u)^\mu$ is $|\mu|\sigma-\gamma_t(\mu)+k_\mu$.

Burgers equation (2):

$$\sigma - 2 = 2\sigma - 1 + 0 \Rightarrow \sigma_c = -1.$$

Example 2 (KdV equation).

$$u_{ttt} - 6uu_t + u_x = 0 \tag{20}$$

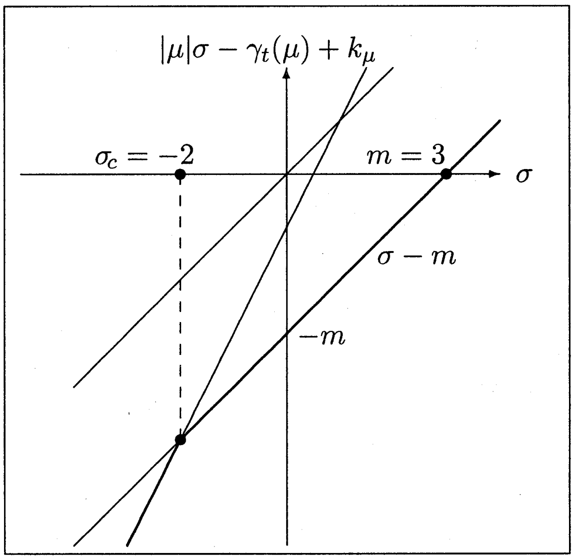
has

$$\sigma - 3 = 2\sigma - 1 + 0 \Rightarrow \sigma_c = -2,$$

and Laurent series solutions

$$u = 2t^{-2} + gt^2 + ht^4 - \frac{1}{24}g_x t^5 + \cdots,$$

where $g = g(x)$ and $h = h(x)$ are arbitrary holomorphic functions.



Characteristic exponent of KdV equation (20)

Lemma 1.
(i) σ_c is invariant with respect to coordinate change which keeps the variable t.
(ii) $\sigma_c \leqslant m_0 \leqslant m - 1$, where m_0 is the order of differentiation with respect to t in $f(t, u; \partial^\Lambda u)$.

3 Singular Solutions

We assume

(A-1) *$f(t, x; Z)$ is a polynomial in Z of degree greater than or equal to* 2.

Under (A-1), the characteristic exponent

$$\sigma_c = \max_{\substack{\mu \in \mathcal{M} \\ |\mu| \geqslant 2}} \frac{\gamma_t(\mu) - m - k_\mu}{|\mu| - 1}, \tag{21}$$

is a rational number strictly less than m_0, and the subset

$$\mathcal{M}^* := \{\mu \in \mathcal{M} : \ |\mu|\sigma_c - \gamma_t(\mu) + k_\mu = \sigma_c - m\}. \tag{22}$$

is not empty. We call the nonlinear term corresponding to μ in $\mathcal{M}^*$ **principal nonlinear term**.

(A-2) *If* $\mu \in \mathcal{M}^*$ *then* $\mu_{j,\alpha} = 0$ *for* $|\alpha| \geqslant 1$

We construct a solution to (16) in the form:

$$u(t,x) := t^{\sigma_c} \sum_{n=0}^{\infty} u_n(x)\, t^{n/p}, \tag{23}$$

where p is the denominator of the reduced fraction σ_c.

Substitute (23) into (16), we obtain recursion equations:

$$\begin{cases} P_c(x; u_0) \cdot u_0 = 0, \\ \\ Q_c\Big(x; u_0; \dfrac{n}{p}\Big) \cdot u_n = R_n\big(x; \partial_x^\alpha u_0, \dots, \partial_x^\alpha u_{n-1}\big)_{|\alpha| \leqslant m}, \end{cases} \tag{24}$$

where

$$P_c(x;\eta) := [\sigma_c;\, m] - \sum_{\mu \in \mathcal{M}^*} f_{\mu,0}(x) \Big(\prod_{(j,\alpha)\in\Lambda} [\sigma_c;\, j]^{\mu_{j,\alpha}} \Big) \eta^{|\mu|-1}, \tag{25}$$

and

$$Q_c(x;\eta;\rho) := [\rho + \sigma_c;\, m] - \sum_{\mu \in \mathcal{M}^*} f_{\mu,0}(x) \times \Big(\prod_{(j,\alpha)\in\Lambda} [\sigma_c;\, j]^{\mu_{j,\alpha}} \Big) \Big(\sum_{(j,\alpha)\in\Lambda} \mu_{j,\alpha} \frac{[\rho+\sigma_c;\, j]}{[\sigma_c;\, j]} \Big) \eta^{|\mu|-1}. \tag{26}$$

Here we have set for $\rho \in \mathbb{R}$ and $j \in \mathbb{N}$,

$$[\rho;\, j] := \rho(\rho-1)\cdots(\rho-j+1). \tag{27}$$

$P_c(x;\eta)$ and $Q_c(x;\eta;\rho)$ are polynomials in η and ρ and depend only on principal nonlinear terms. The order in η is $\max_{\mu\in\mathcal{M}^*} |\mu| - 1$ and m in ρ.

Burgers equation (2):

$$\sigma_c = -1, \qquad P_c(x;\eta) = 2 - 2\eta,$$
$$Q_c(x;\eta = 1;\rho) = (\rho+1)(\rho-2).$$

KdV equation (20):

$$\sigma_c = -2, \quad P_c(x;\eta) = -24 + 12\eta,$$
$$Q_c(x;\eta = 2;\rho) = (\rho+1)(\rho-4)(\rho-6).$$

Remark. If $k_\mu = 0$ for all $\mu \in \mathcal{M}^*$, then we have

$$Q_c(x;\eta;\rho = -1) = \frac{\sigma_c - m}{\sigma_c} P_c(x,\eta). \tag{28}$$

(A-3) *The equation $P_c(x;\eta) = 0$ in η has at least one solution $\eta = u_0(x)$ which is holomorphic in a neighborhood of $x = 0$ and $u_0(x) \not\equiv 0$.*

(A-4) One of the following holds for each $n \geqslant 1$

$$Q_c\Big(0; u_0(0); \frac{n}{p}\Big) \neq 0 \tag{a}$$

$$Q_c\Big(x; u_0(x)\frac{n}{p}\Big) \equiv 0, \qquad R_n(x, \dots \partial_x^\alpha u_0, \dots, \partial_x^\alpha u_{n-1}) \equiv 0 \tag{b}$$

$$\begin{cases} Q_c\Big(0; u_0(0); \frac{n}{p}\Big) = 0, & Q_c\Big(x; u_0(x); \frac{n}{p}\Big) \not\equiv 0 \\ Q_c\Big(x; u_0(x); \frac{n}{p}\Big) \text{ divides } & R_n(x, \dots \partial_x^\alpha u_0, \dots, \partial_x^\alpha u_{n-1}) \end{cases} \tag{c}$$

Remark. In case of (a) or (c), u_n is determined uniquely, and in case of (b), $u_n(x)$ may be any holomorphic function.

Remark.

$$Q_c(0; u_0(0); \rho) = 0$$

has at most m distinct roots.

Theorem 2. *Suppose* (A-1), (A-2), (A-3), (A-4) *are satisfied. Then we can construct a solution to* (16) *in the form* (23). *Moreover all formal solutions* (23) *converge near the origin in $\mathbb{C}_t \times \mathbb{C}_x^d$.*

We apply a theorem by Gérad–Tahara [1] to prove the convergence of formal solutions. For a positive integer N, we put

$$w_N(t,x) := \sum_{n=0}^{\infty} u_{N+n+1}(x)\, t^{\frac{n+1}{p}}.$$

Proposition 1. *If the formal series* (23) *satisfies the equation* (16), *then w_N satisfies the following differential equation:*

$$Q_c\Big(x; u_0(x); t\partial_t + \frac{N}{p}\Big) w_N = t^{1/p} \cdot G\Big(t^{1/p}, x; \big((t\partial_t)^j \partial_x^\alpha w_N\big)_{(j,\alpha)\in\Lambda}\Big), \tag{29}$$

where $G(\tau, x; Z)$ is a polynomial in Z with coefficients holomorphic near the origin in $\mathbb{C}_{\tau,x}^{d+1}$.

Next put $\tau = t^{1/p}$ and

$$\tilde{w}_N(\tau, x) = \sum_{n=0}^{\infty} u_{N+n+1}(x)\tau^{n+1}. \tag{30}$$

Then $\tilde{w}_N(0,x) \equiv 0$, and by using the relation $t\partial_t = \frac{1}{p}\tau\partial_\tau$, we obtain $\tilde{w}_N$ satisfies

$$Q_c\Big(x; u_0(x); \frac{1}{p}\tau\partial_\tau + \frac{N}{p}\Big)\tilde{w}_N = \tau \cdot G\Big(\tau, x; \big((\frac{1}{p}\tau\partial_\tau)^j \partial_x^\alpha \tilde{w}_N\big)_{(j,\alpha)\in\Lambda}\Big). \tag{31}$$

Equation (31) satisfies the conditions (13) and (14), and its characteristic polynomial is

$$C(\rho, x) = Q_c\Big(x; u_0(x); \frac{1}{p}(\rho + N)\Big). \tag{32}$$

If we take N sufficiently large, then $C(n, 0) \neq 0$ for all positive integers.

4 Prolongation of Solutions

We need to define a modified version of characteristic exponent.

Definition 2. For (16), we define σ_c^* by

$$\sigma_c^* = \sup_{\substack{\mu \in \mathcal{M} \\ \nu \leqslant \mu, |\nu| \geqslant 2}} \frac{\gamma_t(\nu) - m - k_\mu}{|\nu| - 1}. \tag{33}$$

Example 3.

$$u_{tt} + 6uu_t^3 + xu_t^2 + uu_x = 0,$$

has a singular solution with exponent $\sigma_c = \dfrac{1}{3}$:

$$u = t^{1/3} - \frac{x}{12} t^{2/3} + \frac{x^2}{240} t^{3/3} + \frac{x^3}{5184} t^{4/3} - \cdots,$$

and ones with exponent $\sigma_c^* = \dfrac{1}{2}$:

$$u = \frac{1}{3} g^2 + \frac{1}{g} t^{1/2} + \left(-\frac{1}{2g^4} - \frac{x}{6g^2}\right) t + \cdots,$$

where $g = g(x)$ is an arbitrary holomorphic function with $g(0) \neq 0$.

Lemma 2.
(i) $\sigma_c \leqslant \sigma_c^* \leqslant m_0 (\leqslant m - 1)$.
(ii) If $\sigma_c \leqslant 0$, *then* $\sigma_c = \sigma_c^*$.

Definition 3. For $\sigma \in \mathbb{R}$, we define $\delta_c(\sigma)$ by

$$\begin{aligned}\delta_c(\sigma) &:= \inf_{\substack{\mu \in \mathcal{M} \\ \nu \leqslant \mu, |\nu| \geqslant 2}} (|\nu| - 1)\sigma - \gamma_t(\nu) + m + k_\mu \\ &= \inf_{\substack{\mu \in \mathcal{M} \\ \nu \leqslant \mu, |\nu| \geqslant 2}} \left(|\nu|\sigma - \gamma_t(\nu) + k_\mu\right) - (\sigma - m).\end{aligned}$$

Lemma 3.
(i) $\delta_c(\sigma) \geqslant 0$ *if and only if* $\sigma \geqslant \sigma_c^*$.
(ii) If $\sigma > \sigma_c^*$, *then* $\delta_c(\sigma) > 0$.
(iii) $\delta_c(m_0) > 0$.
(iv) If $\delta_c(\sigma) > 0$ *and* $\sigma \leqslant m_0$, *then there is a constant* $\delta > 0$ *such that*

$$|\nu|\sigma - \gamma_t(\nu) + k_\mu \geqslant \sigma - m + \delta$$

for all $\mu \in \mathcal{M}$, $\nu \leqslant \mu$.

Definition 4. $u \in \mathcal{O}(\Omega_-)$ is bounded of order σ in Ω_- means that $\exists M > 0$ such that for all $(t, x) \in \Omega_-$, if $\sigma \leqslant 0$

$$|u(t,x)| \leqslant M|\Re t|^{\sigma}$$

or if $\sigma > 0$,

$$|\partial_t^j u(t,x)| \leqslant \begin{cases} M & \text{for } j = 0, 1, .., \lfloor \sigma \rfloor, \\ M|\Re t|^{\sigma - j} & \text{for } j = \lfloor \sigma \rfloor + 1, \end{cases}$$

Example 4. $t^{\sigma} \cdot h(t, x)$ with a holomorphic function $h(t, x)$ is bounded of order σ, and $\log t \cdot h(t, x)$ is bounded of order $-\epsilon$ for any $\epsilon > 0$.

Theorem 3. *If $u \in \mathcal{O}(\Omega_-)$ satisfies Equation* (16) *and is bounded of order σ in Ω_- with $\delta_c(\sigma) > 0$, then u is holomorphic in a neighborhood of the origin. Especially if $\sigma > \sigma_c^*$ or $\sigma = m_0$, then u is holomorphic near the origin.*

Corollary 1. *If $u \in \mathcal{O}(\Omega_-)$ satisfies Equation* (16) *and the derivatives of u up to order m_0 are bounded in Ω_-, then u is holomorphic near the origin.*

Remark 1. Examples 1 and 2 give singular solutions which are bounded of order $\sigma_c = \sigma_c^*$, and Example 3 gives ones of order σ_c^* with $\sigma_c^* > \sigma_c$.

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